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Finslerian-Type GAF Extensions of the Riemannian Framework in Digital Image Processing

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Abstract. Digital image processing was recently proved to be successfully approached by variational tools, which extend the Casseles-Kimmel-Sapiro weighted length problem. Such tools essentially lead to the so-called Geodesic Active Flow (GAF) process, which relies on the derived mean curvature flow PDE. This prolific process is valuable due to both the provided numeric mathematical insight, which requires specific nontrivial choices for implementing the related algorithms, and the variety of possible underlying specific geometric structures. A natural Finsler extension of Randers type was recently developed by the authors, which emphasizes the anisotropy given by the straightforward gradient, while considering a particular scaling of the Lagrangian. The present work develops to its full extent the GAF process to the Randers Finslerian framework: the evolution equations of the model are determined in detail, Matlab simulations illustrate the obtained theoretic results and conclusive remarks are drawn. Finally, open problems regarding the theoretic model and its applicative efficiency are stated.

1. Introduction

Geometric approach in image processing deals with surfaces associated to images. Usually, an image surface is treated as embedded in a Riemannian space. Within this framework, image processing issues are dealt using tools of differential geometry, e.g., the Beltrami flow, which achieves minimization; here, the PDE flow provides surfaces with extremal Polyakov action [16].

The Geodesic Active Field (GAF) technique for image registration, proposed in [17], is based on minimization of the deformation field, which provides an evolutive chain of surfaces. The weighted Polyakov energy functional is minimized in a Riemannian type framework by means of the Beltrami flow [15, 16]. The flow PDE is a directional independent (isotropic) evolutive equation of an 2-dimensional geometric active object [12].

An application of the isotropic curve evolution, as an 1-dimensional geometric active object, appeared for the first time at Caselles-Kimmel-Sapiro, as geodesic active contours (GAC) [7].

The directional dependence of the structural tensor on the embedded space appears in [14], where the anisotropic curve length is considered for curves extraction, and in [11], where the segmentation is considered by the Euclidean metric complemented by the structure tensor that depends on the image gradient. Both cases consider geodesic active contours in the one-dimensional case.

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In [2] the authors suggest a direction-dependent evolution of the embedded image surface, based on 0-homogeneous metric tensor established on the surface instead of the weight function inside the minimized functional. An approximate minimizing flow is presented, containing only the most significant term from the scaled extremal equation.

In this paper we intent to derive the whole anisotropic flow for the established Finslerian structure of Randers type on the image surface.

2. Beltrami framework and Finsler structure

The generalized Beltrami framework with weighted Polyakov action was introduced in [6], considered in [5, 17] and widely applied in computer vision [13, 17–19]. We further consider the embedding map

$$X : D \to M, \quad (\dim D = n < \dim M = m), \tag{1}$$

producing a submanifold $\Sigma = X(D)$ of the Riemannian manifold (M, h) endowed with a Riemannian metric g - which is not necessarily induced from h. Under the consideration is the evolution of the embedded surface (Σ, g) towards the extreme state of minimal Polyakov action $S(X, g, h, f) = \int L \, dx^1 \dots dx^n$, with the Lagrangian density

$$L = f(X, g, h) \cdot g^{\mu\nu} \frac{\partial X^i}{\partial x^\mu} \frac{\partial X^j}{\partial x^\nu} h_{ij} \sqrt{g}, \tag{2}$$

where f is the weight function and g is the determinant of the metric tensor $(g_{\mu\nu})$.

Due to the complex dependencies in L we will use the following brief notation for a given function Φ,

$$\Phi_{;\alpha} := \frac{\partial \Phi}{\partial x^\alpha} \qquad \Phi_{,i} := \frac{\partial \Phi}{\partial X^i} \qquad \Phi_{,\binom{i}{\alpha}} := \frac{\partial \Phi}{\partial X^i_\alpha}.$$

The Euler-Lagrange equation

$$\partial_t X^r = -\frac{1}{2} \frac{1}{\sqrt{\gamma}} h^{ir} \left(L_{,i} - L_{,\binom{i}{\alpha};\alpha} \right) \tag{3}$$

describes the gradient descent flow, called the Beltrami flow,

$$\partial X^r_t = f \tau^r(X) - \frac{n}{2} f_{,i} \, h^{ir} + f_{,i} \, g^{\sigma\mu} X^i_\sigma X^r_\mu \,, \tag{4}$$

where $\tau^r(X)$ is the tension field containing the Laplace-Beltrami operator Δ_g on Σ,

$$\tau^r(X) = \Delta_g(X^r) + g^{\sigma\mu} \Gamma^r_{kl} X^k_\sigma X^l_\mu \,, \tag{5}$$

$$\Delta_g(X^r) = \frac{1}{\sqrt{g}} \partial_\alpha \left(\sqrt{g} g^{\alpha\sigma} X^r_\sigma \right) = g^{\alpha\sigma} X^r_{\alpha\sigma} - g^{\rho\theta} \Gamma^\sigma_{\rho\theta} X^r_\sigma. \tag{6}$$

The choices of the image metric g and of the weight function f are able to produce various particular flows. Two important subcases are:

- mean curvature flow, with induced g and $f \equiv 1$,
- tension flow, with arbitrary $g = g(x)$ and $f \equiv 1$.

The tension flow deforms the embedding X to a harmonic map, an extremal when the Polyakov action is viewed as an energy functional in $C^\infty(D, M)$ [10]. The Finslerian extension will produce the anisotropic Beltrami flow on the tangent bundle of the image surface.

The Finsler structure is given on a differential manifold by a metric tensor field derived from a fundamental function F defined over its tangent bundle with certain particular requirements [3, 4, 9]. The produced metric is a d-tensor on the tangent bundle [3, 9].

To distinguish a directionally dependent metric $\gamma = \gamma_{\mu\nu} dx^\mu \otimes dx^\nu$ on Σ from the induced metric denoted with g, we will use the notation with Greek letters.

The Finsler norm (fundamental function) on the image surface Σ is a mapping $F : T\Sigma \to \mathbb{R}$, satisfying the following conditions:

- (smoothness) F is continuous on $T\Sigma$ and is C^∞ on the slit tangent space $\widetilde{T\Sigma} = \{(x, y) \in T\Sigma \mid y \neq 0\}$;
- (positivity) $F(u) \geq 0, \forall u \in T\Sigma$;
- (positive 1-homogeneity)

$$F(x, ty) = tF(x, y), \ \ \forall t \in (0, \infty), \ \forall (x, y) \in T\Sigma;$$

- (strong convexity) for any $u = (x, y) \in T\Sigma$, the bilinear form $\gamma_u : T_x\Sigma \times T_x\Sigma \to \mathbb{R}$, given by

$$\gamma_u(v, w) = \frac{1}{2}(Hess_y F^2)(v, w) \quad \forall v, w \in T_x\Sigma, \quad u = (x, y) \in T\Sigma,$$

  is positive definite.

If the Finsler norm is a linear deformation of a norm produced by the Riemannian metric $\alpha = a_{\mu\nu} dx^\mu \otimes dx^\nu$,

$$F(x, v) = \sqrt{a_{\mu\nu}(x) v^\mu v^\nu} + b_\mu v^\mu,$$

then under certain constrains on the 1-form $\beta = b_\mu dx^\mu$, the produced metric is of Randers type [3, 8].

In the following we shall use a general deformation $\gamma_{\mu\nu}$ of the induced metric $g_{\mu\nu}$:

$$\gamma_{\mu\nu} = g_{\mu\nu} + \varphi_{\mu\nu} \tag{7}$$

$$\varphi_{\mu\nu} = A g_{\mu\nu} + B g_{\mu\alpha} g_{\nu\beta} v^\alpha v^\beta + C g_{\mu\alpha} v^\alpha b_\nu + C g_{\nu\beta} v^\beta b_\mu + b_\mu b_\nu, \tag{8}$$

where

$$G = \sqrt{g_{\alpha\beta} v^\alpha v^\beta}, \quad \Omega = b_\alpha v^\alpha, \quad A = \frac{\Omega}{G}, \quad B = -\frac{\Omega}{G^3}, \quad C = \frac{1}{G}$$

and $\gamma = \det(\gamma_{\mu\nu}) = (\frac{F}{G})^3 g$ is the determinant value. The associated dual metric tensor has the components $\gamma^{\mu\nu} = g^{\mu\nu} + \rho^{\mu\nu}$, where

$$\rho^{\mu\nu} = -\frac{\Omega}{F} g^{\mu\nu} + \frac{\Omega + G g^{\alpha\beta} b_\alpha b_\beta}{F^3} v^\nu v^\mu - \frac{G}{F^2} \left(g^{\mu\alpha} b_\alpha v^\nu + g^{\nu\alpha} b_\alpha v^\mu \right).$$

Due to the computational complexity of the components of the deformed metric tensor (7), it is more convenient not to specify at this step the exact form of the deformation field $\varphi_{\mu\nu}$, still assuming as minimal requirements that γ should be symmetric, regular and of constant signature [4].

3. Finslerian extension of the Beltrami framework

In order to develop the anisotropic Beltrami framework, we consider, as above, the embedding X and fix the Riemannian metric of the embedding space (M, h_{ij}). We further extend in additive manner the induced metric $g = g_{\sigma\mu}(x) dx^\sigma \otimes dx^\mu$ to the new deformed anisotropic metric, $\gamma = \gamma_{\sigma\mu}(x, v) dx^\sigma \otimes dx^\mu$:

$$\gamma_{\mu\nu}(x, v) = g_{\mu\nu}(x) + a \cdot \varphi_{\mu\nu}(x, v). \tag{9}$$

The anisotropic tensor $\varphi_{\mu\nu}$ will be regarded as the additional tensor, and obviously, $a = 0$ leads to the classic Beltrami framework.

The Polyakov action to be minimized has the same form, but depends on direction,

$$E(X, \gamma_{\mu\nu}, h_{jk}) = \int f \gamma^{\mu\nu} X^i_\mu X^j_\nu h_{ij} \sqrt{\gamma}\, dx^1 \dots dx^n.$$

The Euler-Lagrange equation which produces the flow will be derived in accordance with the Hilbert-Palatini variational principle [1].

Theorem 3.1. *(The anisotropic weighted Beltrami flow) The PDE of the descent flow which provides the minimality of the weighted Polyakov action on the surface* (Σ, γ) *- which is embedded into the Riemannian manifold* (M, h) *by the mapping (1), is*

$$\begin{aligned} \partial_t X^r = {} & \frac{1}{2} f_{,\binom{i}{\alpha};\alpha} \gamma^{\sigma\mu} g_{\sigma\mu} h^{ir} - \frac{1}{2} f_{,i} \gamma^{\sigma\mu} g_{\sigma\mu} h^{ir} \\ & + \frac{1}{2} f_{,\binom{i}{\alpha}} h^{ir} \left[(\gamma^{\sigma\mu})_{;\alpha} g_{\sigma\mu} + \gamma^{\sigma\mu} g_{\sigma\mu;\alpha} + \gamma^{\sigma\mu} g_{\sigma\mu} (\ln \sqrt{\gamma})_{;\alpha} \right] \\ & + \frac{1}{2} f_{;\alpha} h^{ir} \left[(\gamma^{\sigma\mu})_{,\binom{i}{\alpha}} g_{\sigma\mu} + \gamma^{\sigma\mu} g_{\sigma\mu,\binom{i}{\alpha}} + \gamma^{\sigma\mu} g_{\sigma\mu} (\ln \sqrt{\gamma})_{,\binom{i}{\alpha}} \right] \\ & + f \tau^r(X) \\ & + \frac{1}{2} h^{ir} f \Big\{ g_{\sigma\mu;\alpha} \left[(\gamma^{\sigma\mu})_{,\binom{i}{\alpha}} + \gamma^{\sigma\mu} (\ln \sqrt{\gamma})_{,\binom{i}{\alpha}} \right] \\ & \qquad + g_{\sigma\mu} \Big[(\gamma^{\sigma\mu})_{,\binom{i}{\alpha};\alpha} + (\gamma^{\sigma\mu})_{,\binom{i}{\alpha}} (\ln \sqrt{\gamma})_{;\alpha} + (\gamma^{\sigma\mu})_{;\alpha} (\ln \sqrt{\gamma})_{,\binom{i}{\alpha}} \\ & \qquad\qquad + \gamma^{\sigma\mu} \tfrac{1}{\sqrt{\gamma}} (\sqrt{\gamma})_{,\binom{i}{\alpha};\alpha} - (\gamma^{\sigma\mu})_{,i} - \gamma^{\sigma\mu} (\ln \sqrt{\gamma})_{,i} \Big] \Big\}, \end{aligned} \tag{10}$$

where $\tau(X)$ *is the tension of the embedding* X *(5) and* $g_{\sigma\mu} = h_{kl} X^k_\sigma X^l_\mu$ *is the induced metric tensor field on the embedded surface* Σ.

Proof. A straightforward calculation expresses the partial derivatives of the Lagrangian density by the partial derivatives of both metrics, the embedding and the weight function, as follows:

$$L_{,i} = f_{,i}\, \gamma^{\sigma\mu} g_{\sigma\mu} \sqrt{\gamma} + f (\gamma^{\sigma\mu})_{,i}\, g_{\sigma\mu} \sqrt{\gamma} + \gamma^{\sigma\mu} g_{\sigma\mu,i} \sqrt{\gamma} + \gamma^{\sigma\mu} g_{\sigma\mu} (\sqrt{\gamma})_{,i}, \tag{11}$$

$$\begin{aligned} L_{,\binom{i}{\alpha}} = {} & f_{,\binom{i}{\alpha}}\, \gamma^{\sigma\mu} g_{\sigma\mu} \sqrt{\gamma} \\ & + f \left((\gamma^{\sigma\mu})_{,\binom{i}{\alpha}} g_{\sigma\mu} \sqrt{\gamma} + \gamma^{\sigma\mu} g_{\sigma\mu,\binom{i}{\alpha}} \sqrt{\gamma} + \gamma^{\sigma\mu} g_{\sigma\mu} (\sqrt{\gamma})_{,\binom{i}{\alpha}} \right), \end{aligned} \tag{12}$$

$$\begin{aligned} L_{,\binom{i}{\alpha};\alpha} = {} & f_{,\binom{i}{\alpha};\alpha} \gamma^{\sigma\mu} g_{\sigma\mu} \sqrt{\gamma} \\ & + f_{,\binom{i}{\alpha}} \left[(\gamma^{\sigma\mu})_{;\alpha} g_{\sigma\mu} \sqrt{\gamma} + \gamma^{\sigma\mu} g_{\sigma\mu;\alpha} \sqrt{\gamma} + \gamma^{\sigma\mu} g_{\sigma\mu} (\sqrt{\gamma})_{;\alpha} \right] \\ & + f_{;\alpha} \left[(\gamma^{\sigma\mu})_{,\binom{i}{\alpha}} g_{\sigma\mu} \sqrt{\gamma} + \gamma^{\sigma\mu} g_{\sigma\mu,\binom{i}{\alpha}} \sqrt{\gamma} + \gamma^{\sigma\mu} g_{\sigma\mu} (\sqrt{\gamma})_{,\binom{i}{\alpha}} \right] \\ & + f \Big\{ (\gamma^{\sigma\mu})_{,\binom{i}{\alpha};\alpha} g_{\sigma\mu} \sqrt{\gamma} + (\gamma^{\sigma\mu})_{,\binom{i}{\alpha}} g_{\sigma\mu;\alpha} \sqrt{\gamma} + (\gamma^{\sigma\mu})_{,\binom{i}{\alpha}} g_{\sigma\mu} (\sqrt{\gamma})_{;\alpha} \\ & \quad + (\gamma^{\sigma\mu})_{;\alpha} g_{\sigma\mu,\binom{i}{\alpha}} \sqrt{\gamma} + \gamma^{\sigma\mu} g_{\sigma\mu,\binom{i}{\alpha};\alpha} \sqrt{\gamma} + \gamma^{\sigma\mu} g_{\sigma\mu,\binom{i}{\alpha}} (\sqrt{\gamma})_{;\alpha} \\ & \quad + (\gamma^{\sigma\mu})_{;\alpha} g_{\sigma\mu} (\sqrt{\gamma})_{,\binom{i}{\alpha}} + \gamma^{\sigma\mu} g_{\sigma\mu;\alpha} (\sqrt{\gamma})_{,\binom{i}{\alpha}} + \gamma^{\sigma\mu} g_{\sigma\mu} (\sqrt{\gamma})_{,\binom{i}{\alpha};\alpha} \Big\}. \end{aligned} \tag{13}$$

Plugging (11) and (13) into (3), and using the abbreviate notation

$$\frac{1}{\sqrt{\gamma}} (\sqrt{\gamma})_\star = \left(\ln \sqrt{\gamma} \right)_\star$$

for all kind of derivatives $\star$: $_{;\alpha}$, $_{,i}$, $_{,\binom{i}{\alpha}}$, produces the gradient descent flow for the surface embedded in the Riemannian space in the form (10). □

In the following, we propose the replacement of the weighted Polyakov action with the non-weighted one, but considering on the surface a more general metric, of anisotropic (direction-dependent) type. Hence, for $f \equiv 1$, one proves immediately the following result.

Theorem 3.2. *(The anisotropic Beltrami flow) The explicit form of the anisotropic Beltrami flow minimizing non-weighted Polyakov action with generalized Lagrange image metric γ is*

$$\begin{aligned}\partial_t X^r = \;& \tau^r(X) \\ &+\frac{1}{2}h^{ir}\Big\{g_{\sigma\mu;\alpha}\Big[(\gamma^{\sigma\mu})_{,\binom{i}{\alpha}} + \gamma^{\sigma\mu}(\ln\sqrt{\gamma})_{,\binom{i}{\alpha}}\Big] \\ &\quad+g_{\sigma\mu}\Big[(\gamma^{\sigma\mu})_{,\binom{i}{\alpha};\alpha} + (\gamma^{\sigma\mu})_{,\binom{i}{\alpha}}(\ln\sqrt{\gamma})_{;\alpha} + (\gamma^{\sigma\mu})_{;\alpha}(\ln\sqrt{\gamma})_{,\binom{i}{\alpha}} \\ &\qquad+\gamma^{\sigma\mu}\frac{1}{\sqrt{\gamma}}(\sqrt{\gamma})_{,\binom{i}{\alpha};\alpha} - (\gamma^{\sigma\mu})_{,i} - \gamma^{\sigma\mu}(\ln\sqrt{\gamma})_{,i}\Big]\Big\}.\end{aligned}$$

The adjustment of the previous general result to the case where the image metric is of type $\gamma = g + \varphi$, will be achieved by using the following auxiliary results on the derivatives of the contravariant tensor and on the determinant expressed in terms of the covariant metric components.

Lemma 3.3. *The variations of the determinant and of the dual metric tensor for γ are described by the following expressions*

$$\begin{aligned}\gamma_* &= \gamma\gamma^{\lambda\tau}\gamma_{\lambda\tau*}, \qquad (\ln\sqrt{\gamma})_* = \frac{1}{2}\gamma^{\lambda\tau}\gamma_{\lambda\tau*} \\ (\gamma^{\sigma\mu})_* &= -\gamma^{\sigma\lambda}\gamma^{\mu\tau}\gamma_{\lambda\tau*} \\ (\gamma^{\sigma\mu})_{,\binom{i}{\alpha};\alpha} &= \left(\gamma^{\sigma\rho}\gamma^{\lambda\theta}\gamma^{\mu\tau} + \gamma^{\sigma\lambda}\gamma^{\mu\rho}\gamma^{\tau\theta}\right)\gamma_{\rho\theta;\alpha}\gamma_{\lambda\tau,\binom{i}{\alpha}} - \gamma^{\sigma\lambda}\gamma^{\mu\tau}\gamma_{\lambda\tau,\binom{i}{\alpha};\alpha} \\ \frac{1}{\sqrt{\gamma}}(\sqrt{\gamma})_{,\binom{i}{\alpha};\alpha} &= \frac{1}{2}\left(\frac{1}{2}\gamma^{\rho\theta}\gamma^{\lambda\tau} - \gamma^{\lambda\rho}\gamma^{\tau\theta}\right)\gamma_{\rho\theta;\alpha}\gamma_{\lambda\tau,\binom{i}{\alpha}} + \frac{1}{2}\gamma^{\lambda\tau}\gamma_{\lambda\tau,\binom{i}{\alpha};\alpha},\end{aligned}$$

where $\gamma_{\mu\nu*} = g_{\mu\nu*} + \varphi_{\mu\nu*}$ and ϕ_* stands for $\phi_{;\alpha}$, $\phi_{,i}$ and $\phi_{,\binom{i}{\alpha}}$.

Proof. The assertion is obtained by straightforward calculations. □

Theorem 3.4. *The anisotropic Beltrami flow of an image surface with metric tensor* (9) *is described by the evolution PDE:*

$$\begin{aligned}\partial_t^{GL} X^r = \;& \tau^r(X) \\ &+\frac{1}{2}h^{ir}\left\{\frac{1}{2}\gamma^{\sigma\mu}\gamma^{\lambda\tau} - \gamma^{\sigma\lambda}\gamma^{\mu\tau}\right\}\cdot \\ &\quad\Big\{g_{\sigma\mu;\alpha}g_{\lambda\tau,\binom{i}{\alpha}} + g_{\sigma\mu}g_{\lambda\tau,\binom{i}{\alpha};\alpha} - g_{\sigma\mu}g_{\lambda\tau,i} + g_{\sigma\mu;\alpha}\varphi_{\lambda\tau,\binom{i}{\alpha}} + g_{\sigma\mu}\varphi_{\lambda\tau,\binom{i}{\alpha};\alpha} - g_{\sigma\mu}\varphi_{\lambda\tau,i}\Big\} \\ &+\frac{1}{2}h^{ir}g_{\sigma\mu}\left\{g_{\rho\theta;\alpha}g_{\lambda\tau,\binom{i}{\alpha}} + g_{\rho\theta;\alpha}\varphi_{\lambda\tau,\binom{i}{\alpha}} + g_{\lambda\tau,\binom{i}{\alpha}}\varphi_{\rho\theta;\alpha} + \varphi_{\rho\theta;\alpha}\varphi_{\lambda\tau,\binom{i}{\alpha}}\right\}\cdot \\ &\quad\Big\{\gamma^{\sigma\rho}(\gamma^{\lambda\theta}\gamma^{\mu\tau} - \frac{1}{2}\gamma^{\mu\theta}\gamma^{\lambda\tau}) + \gamma^{\sigma\lambda}(\gamma^{\mu\rho}\gamma^{\tau\theta} - \frac{1}{2}\gamma^{\mu\tau}\gamma^{\rho\theta}) - \frac{1}{2}\gamma^{\sigma\mu}(\gamma^{\lambda\rho}\gamma^{\tau\theta} - \frac{1}{2}\gamma^{\rho\theta}\gamma^{\lambda\tau})\Big\}.\end{aligned} \tag{14}$$

Lemma 3.5. *The derivatives of the components of the additional Randers tensor* (8) *in the anisotropic Beltrami framework are*

$$\begin{aligned}\varphi_{\sigma\mu\star} = & \frac{1}{G^3}\left[(\Omega_\star V - \frac{1}{2}\Omega V_\star)g_{\sigma\mu} + \Omega V g_{\sigma\mu\star}\right] + b_{\sigma\star}b_\mu + b_\sigma b_{\mu\star} \\ & + \frac{1}{G^5}\left[(\frac{3}{2}\Omega V_\star - \Omega_\star V)g_{\sigma\rho}g_{\mu\theta} - \Omega V(g_{\sigma\rho\star}g_{\mu\theta} + g_{\sigma\rho}g_{\mu\theta\star})\right]v^\rho v^\theta \\ & + \frac{1}{G^3}\left[V(g_{\sigma\rho\star}b_\mu + g_{\sigma\rho}b_{\mu\star} + g_{\mu\rho\star}b_\sigma + g_{\mu\rho}b_{\sigma\star}) - V_\star(g_{\sigma\rho}b_\mu + g_{\mu\rho}b_\sigma)\right]v^\rho\end{aligned} \tag{15}$$

The mixed derivatives of the additional tensor components $\varphi_{\sigma\mu}$ in the Randers type metric on the image surface are

$$\begin{aligned}\varphi_{\sigma\mu,\binom{i}{\alpha};\alpha} = & \frac{1}{G^5}\left[\left(\Omega_{,\binom{i}{\alpha};\alpha}V^2 - \frac{1}{2}\Omega_{,\binom{i}{\alpha}}V_{;\alpha}V - \frac{1}{2}\Omega_{;\alpha}V_{,\binom{i}{\alpha}}V - \frac{1}{2}\Omega V_{,\binom{i}{\alpha};\alpha}V + \frac{3}{4}\Omega V_{,\binom{i}{\alpha}}V_{;\alpha}\right)g_{\sigma\mu}\right. \\ & \left. + V\left(\Omega_{,\binom{i}{\alpha}}V - \frac{1}{2}\Omega V_{,\binom{i}{\alpha}}\right)g_{\sigma\mu;\alpha} + V\left(\Omega_{;\alpha}V - \frac{1}{2}\Omega V_{;\alpha}\right)g_{\sigma\mu,\binom{i}{\alpha}} + \Omega V^2 g_{\sigma\mu,\binom{i}{\alpha};\alpha}\right] \\ & + \frac{1}{G^7}\left[\left(-\Omega_{,\binom{i}{\alpha};\alpha}V^2 + \frac{3}{2}\Omega_{,\binom{i}{\alpha}}V_{;\alpha}V + \frac{3}{2}\Omega_{;\alpha}V_{,\binom{i}{\alpha}}V + \frac{3}{2}\Omega V_{,\binom{i}{\alpha};\alpha}V - \frac{15}{4}\Omega V_{,\binom{i}{\alpha}}V_{;\alpha}\right)g_{\sigma\rho}g_{\mu\theta}\right. \\ & + V\left(\frac{3}{2}\Omega V_{,\binom{i}{\alpha}} - \Omega_{,\binom{i}{\alpha}}V\right)\left(g_{\sigma\rho;\alpha}g_{\mu\theta} + g_{\sigma\rho}g_{\mu\theta;\alpha}\right) \\ & + V\left(\frac{3}{2}\Omega V_{;\alpha} - \Omega_{;\alpha}V\right)\left(g_{\sigma\rho,\binom{i}{\alpha}}g_{\mu\theta} + g_{\sigma\rho}g_{\mu\theta,\binom{i}{\alpha}}\right) \\ & \left. - \Omega V^2\left(g_{\sigma\rho,\binom{i}{\alpha};\alpha}g_{\mu\theta} + g_{\sigma\rho,\binom{i}{\alpha}}g_{\mu\theta;\alpha} + g_{\sigma\rho;\alpha}g_{\mu\theta,\binom{i}{\alpha}} + g_{\sigma\rho}g_{\mu\theta,\binom{i}{\alpha};\alpha}\right)\right]v^\rho v^\theta \\ & + \frac{1}{G^5}\left[\left(\frac{3}{4}V_{;\alpha}V_{,\binom{i}{\alpha}} - \frac{1}{2}VV_{,\binom{i}{\alpha};\alpha}\right)\left(g_{\sigma\rho}b_\mu + g_{\mu\rho}b_\sigma\right)\right. \\ & - VV_{,\binom{i}{\alpha}}\left(g_{\sigma\rho;\alpha}b_\mu + g_{\sigma\rho}b_{\mu;\alpha} + g_{\mu\rho;\alpha}b_\sigma + g_{\mu\rho}b_{\sigma;\alpha}\right) \\ & - VV_{;\alpha}\left(g_{\sigma\rho,\binom{i}{\alpha}}b_\mu + g_{\sigma\rho}b_{\mu,\binom{i}{\alpha}} + g_{\mu\rho,\binom{i}{\alpha}}b_\sigma + g_{\mu\rho}b_{\sigma,\binom{i}{\alpha}}\right) \\ & + V^2\left(g_{\sigma\rho,\binom{i}{\alpha};\alpha}b_\mu + g_{\sigma\rho,\binom{i}{\alpha}}b_{\mu;\alpha} + g_{\sigma\rho;\alpha}b_{\mu,\binom{i}{\alpha}} + g_{\sigma\rho}b_{\mu,\binom{i}{\alpha};\alpha}\right. \\ & \left.\left. + g_{\mu\rho,\binom{i}{\alpha};\alpha}b_\sigma + g_{\mu\rho,\binom{i}{\alpha}}b_{\sigma;\alpha} + g_{\mu\rho;\alpha}b_{\sigma,\binom{i}{\alpha}} + g_{\mu\rho}b_{\sigma,\binom{i}{\alpha};\alpha}\right)\right]v^\rho \\ & + b_{\sigma,\binom{i}{\alpha};\alpha}b_\mu + b_{\sigma,\binom{i}{\alpha}}b_{\mu;\alpha} + b_{\sigma;\alpha}b_{\mu,\binom{i}{\alpha}} + b_\sigma b_{\mu,\binom{i}{\alpha};\alpha}.\end{aligned} \tag{16}$$

Proof. By the use of Ω and $V = G^2 = g_{\alpha\beta}v^\alpha v^\beta$, we obtain the variations of the main scalar functions of the linear part of the Randers metric tensor:

$$A_\star = \frac{1}{GV}\left(\Omega_\star V - \frac{1}{2}\Omega V_\star\right), \tag{17}$$

$$B_\star = \frac{1}{GV^2}\left(\frac{3}{2}\Omega V_\star - \Omega_\star V\right), \tag{18}$$

$$C_\star = -\frac{1}{2GV}V_\star, \tag{19}$$

$$A_{,\binom{i}{\alpha};\alpha} = \frac{1}{G^5}\left(\Omega_{,\binom{i}{\alpha};\alpha}V^2 - \frac{1}{2}\Omega_{,\binom{i}{\alpha}}V_{;\alpha}V - \frac{1}{2}\Omega_{;\alpha}V_{,\binom{i}{\alpha}}V - \frac{1}{2}\Omega V_{,\binom{i}{\alpha};\alpha}V + \frac{3}{4}\Omega V_{,\binom{i}{\alpha}}V_{;\alpha}\right), \tag{20}$$

$$B_{,\binom{i}{\alpha};\alpha} = \frac{1}{G^7}\left(-\Omega_{,\binom{i}{\alpha};\alpha}V^2 + \frac{3}{2}\Omega_{,\binom{i}{\alpha}}V_{;\alpha}V + \frac{3}{2}\Omega_{;\alpha}V_{,\binom{i}{\alpha}}V + \frac{3}{2}\Omega V_{,\binom{i}{\alpha};\alpha}V\right) \tag{21}$$

$$C_{,\binom{i}{\alpha};\alpha} = \frac{1}{G^5}\left(\frac{3}{4}V_{;\alpha}V_{,\binom{i}{\alpha}} - \frac{1}{2}VV_{,\binom{i}{\alpha};\alpha}\right). \tag{22}$$

The Leibniz rule and the linearity of derivatives produce

$$\begin{aligned}\varphi_{\sigma\mu\star} = \; & A_\star g_{\sigma\mu} + A g_{\sigma\mu\star} \\ & + \left[B_\star g_{\sigma\rho} g_{\mu\theta} + B g_{\sigma\rho\star} g_{\mu\theta} + B g_{\sigma\rho\star} g_{\mu\theta}\right] v^\rho v^\theta \\ & + \left[C_\star (g_{\sigma\rho} b_\mu + g_{\mu\rho} b_\sigma) + C(g_{\sigma\rho\star} b_\mu + g_{\sigma\rho} b_{\mu\star} + g_{\mu\rho\star} b_\sigma + g_{\mu\rho} b_{\sigma\star})\right] v^\rho \\ & + b_{\sigma\star} b_\mu + b_\sigma b_{\mu\star}.\end{aligned}$$

Substitution of the scalar functions derivatives (17)-(19) leads to (15).

Taking the derivative of the previous equation with respect to the parameter x^α, while keeping $\star =, \binom{i}{\alpha}$ fixed, yields

$$\begin{aligned}\varphi_{\sigma\mu,\binom{i}{\alpha};\alpha} = \; & A_{,\binom{i}{\alpha};\alpha} g_{\sigma\mu} + A_{,\binom{i}{\alpha}} g_{\sigma\mu;\alpha} + A_{;\alpha} g_{\sigma\mu,\binom{i}{\alpha}} + A g_{\sigma\mu,\binom{i}{\alpha};\alpha} \\ & + \Big[B_{,\binom{i}{\alpha};\alpha} g_{\sigma\rho} g_{\mu\theta} + B_{,\binom{i}{\alpha}} \left(g_{\sigma\rho;\alpha} g_{\mu\theta} + g_{\sigma\rho} g_{\mu\theta;\alpha}\right) + B_{;\alpha}\left(g_{\sigma\rho,\binom{i}{\alpha}} g_{\mu\theta} + g_{\sigma\rho} g_{\mu\theta,\binom{i}{\alpha}}\right) \\ & \quad + B\left(g_{\sigma\rho,\binom{i}{\alpha};\alpha} g_{\mu\theta} + g_{\sigma\rho,\binom{i}{\alpha}} g_{\mu\theta;\alpha} + g_{\sigma\rho;\alpha} g_{\mu\theta,\binom{i}{\alpha}} + g_{\sigma\rho} g_{\mu\theta,\binom{i}{\alpha};\alpha}\right)\Big] v^\rho v^\theta \\ & + \Big[C_{,\binom{i}{\alpha};\alpha}\left(g_{\sigma\rho} b_\mu + g_{\mu\rho} b_\sigma\right) + C_{,\binom{i}{\alpha}}\left(g_{\sigma\rho;\alpha} b_\mu + g_{\sigma\rho} b_{\mu;\alpha} + g_{\mu\rho;\alpha} b_\sigma + g_{\mu\rho} b_{\sigma;\alpha}\right) \\ & \quad + C_{;\alpha}\left(g_{\sigma\rho,\binom{i}{\alpha}} b_\mu + g_{\sigma\rho} b_{\mu,\binom{i}{\alpha}} + g_{\mu\rho,\binom{i}{\alpha}} b_\sigma + g_{\mu\rho} b_{\sigma,\binom{i}{\alpha}}\right) \\ & \quad + C\Big(g_{\sigma\rho,\binom{i}{\alpha};\alpha} b_\mu + g_{\sigma\rho,\binom{i}{\alpha}} b_{\mu;\alpha} + g_{\sigma\rho;\alpha} b_{\mu,\binom{i}{\alpha}} + g_{\sigma\rho} b_{\mu,\binom{i}{\alpha};\alpha} \\ & \qquad + g_{\mu\rho,\binom{i}{\alpha};\alpha} b_\sigma + g_{\mu\rho,\binom{i}{\alpha}} b_{\sigma;\alpha} + g_{\mu\rho;\alpha} b_{\sigma,\binom{i}{\alpha}} + g_{\mu\rho} b_{\sigma,\binom{i}{\alpha};\alpha}\Big)\Big] v^\rho \\ & + b_{\sigma,\binom{i}{\alpha};\alpha} b_\mu + b_{\sigma,\binom{i}{\alpha}} b_{\mu;\alpha} + b_{\sigma;\alpha} b_{\mu,\binom{i}{\alpha}} + b_\sigma b_{\mu,\binom{i}{\alpha};\alpha}.\end{aligned}$$

Further, by using the expressions (20)-(22), one obtains (16). □

The general Randers flow, denoted by $\partial_t^{GR} X^r$ is readily obtained as the combination of previous two results.

4. The Finslerian-Randers extension

We shall further consider the Beltrami framework commonly used in image processing, a Monge surface in the 3-dimensional Euclidean space $X : D \to M$, where $(M, h) = (\mathbb{R}^3, h_{ij} = \mathrm{diag}(1, 1, \beta^2))$,

$$X : (x^1, x^2) \mapsto (x^1, x^2, I(x^1, x^2)).$$

The components of the induced metric on the surface are

$$(g_{\mu\nu}) = \begin{pmatrix} 1 + \beta^2 I_{x^1}^2 & \beta^2 I_{x^1} I_{x^2} \\ \beta^2 I_{x^1} I_{x^2} & 1 + \beta^2 I_{x^2}^2 \end{pmatrix},$$

where, for brevity, we use the notation $I_{x^\mu} = \frac{\partial I}{\partial x^\mu}$. The gradient vector field $gradI = (g^{1\alpha} I_{x^\alpha}, g^{2\alpha} I_{x^\alpha})$ naturally produces the following Finslerian norm of Randers type:

$$F(x, v) = \sqrt{g_{\mu\nu} v^\mu v^\nu} + a \cdot v^\alpha I_{x^\alpha}. \tag{23}$$

The resulting Randers metric γ is of Ingarden type [3]. The terms involved in the Ingarden norm are $G(v) = \sqrt{V(v)}$ and $\Omega(v)$, functionals evaluated by

$$\begin{aligned} & V(v) = (1 + \beta^2 I_{x^1}^2)(v^1)^2 + 2\beta^2 I_{x^1} I_{x^2} v^1 v^2 + (1 + \beta^2 I_{x^2}^2)(v^2)^2, \\ & \Omega(v) = a I_{x^1} v^1 + a I_{x^2} v^2, \quad \text{i.e.,} \quad b_\mu = a I_{x^\mu}\end{aligned}$$

for a certain deviation vector $v = (v^1, v^2) \in T_x\Sigma$. Further, the Finslerian-Randers type metric components and the PDE of the evolution flow can be calculated, and only the third component of the flow, $\partial X_t^3 = \partial I_t$ is nontrivial.

5. Discretization and tentative application in image processing

A grayscale image is viewed as a matrix I, whose elements $I(i, j)$ are in correspondence with the locations of the pixels of the image $(i, j) = (x^1, x^2) =: x$ and their values $I(i, j) \in \{0, 1, \ldots, 255\}$, which represent the level of their grey color intensity [1)]. The considered tangent vectors point to the neighboring pixels, $v = (v^1, v^2) \in \{(-1, -1), (-1, 0), (-1, 1), (0, -1), (0, 1), (1, -1), (1, 0), (1, 1)\}$[2)]. The gradient vector is discretized by the shift tangent vector computed as max-abs of the shifts towards the pixels of the eight neighbors of the current pixel.

The partial derivatives of the feature are respectively determined by:

$$\begin{aligned} &I_{x^1}(i, j) = I(i+1, j) - I(i, j), \quad I_{x^2}(i, j) = I(i, j+1) - I(i, j), \\ &I_{x^1x^1}(i, j) = I(i+2, j) + I(i, j) - 2I(i+1, j), \\ &I_{x^2x^2}(i, j) = I(i, j+2) + I(i, j) - 2I(i, j+1), \\ &I_{x^1x^2}(i, j) = I(i+1, j+1) + I(i, j) - I(i+1, j) - I(i, j+1). \end{aligned}$$

The Beltrami induced successive evolutive shifting of the monochrome image $\Sigma = (I(i, j))$ is achieved by Matlab implementation, where each iteration implies the following steps:

- accessing pixels (apart from the boundary) to get the corresponding feature value, I;
- determining the shift tangent vector;
- applying the flow expression on the feature value and the shift tangent vector to obtain the shift value, $\triangle I(i, j)$;
- computing the modified feature value, $I \to I + \triangle I$.

6. Conclusions and perspectives

The classical Beltrami framework and the basics of Finslerian and generalized Lagrange anisotropic structures and of the corresponding anisotropic evolution flows are presented. The Matlab implementation on a 2-dimensional image surface is accomplished for the Ingarden evolution flow.

Based on the implementation results, we are able to conclude that the anisotropic evolution process is substantially slower than the isotropic one, but it is more sensitive. The anisotropic metric structure causes the dependence of the Finsler metric coefficients on the shift vector, and emphasizes the action of the flow at noisy pixels. The output differs from the input by a slight increase of contrast between compact regions.

Further research will address the following specific issues: the selection of an appropriate adaptive direction-dependent structure (possibly non-smooth); the considering of a weight function in order to accelerate the convergence speed of the evolution process; the exploring of the possibilities to apply the anisotropic flow to image processing: feature detection, image enhancement, anti-aliasing and detection of contours.

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1) Alternatively, instead of pixel intensity, processing may be applied to other characteristics of pixels or of their neighborhoods.

2) While processing the image, the corresponding incomplete border stripes will be discarded.

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